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ON THE TRANSFORMATION OF A MONOCYCLIC INTO A RECURRENT SEQUENCE

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1. Let there be given a sequence

$$\xi(0),\ \xi(1), \ldots,\ \xi(i), \ldots \tag{1}$$

of n-stage binary numbers; that is, numbers of the form

$$\xi(i) = \sum_{k=1}^{n} \xi_k(i) \cdot 2^{n_0-k}, \tag{2}$$

where n_0 is any fixed integer, and n is a positive integer.

It will be said that the sequence

$$\eta(0),\ \eta(1), \ldots,\ \eta(i), \ldots \tag{3}$$

of N-stage binary numbers, where $N \geqq n$, contains the sequence (1), if the binary digits of every term $\xi(i)$ of the sequence (1) coincide with the corresponding digits of the corresponding term $\eta(i)$ of the sequence (3); that is, if

$$\eta(i) = \sum_{k=1}^{n} \xi_k(i) \cdot 2^{n_0-k} + \sum_{k=n+1}^{N} \eta_k \cdot 2^{n_0-k}. \tag{4}$$

The transition from the sequence (1) to any sequence (3) containing it will be called an enlargement or extention of the sequence (1), and conversely the transition from the sequence (3) to any sequence (1) contained in it will be called a contraction or restriction of the sequence (3).

2. Any sequence of the form

$$\xi(0), \ldots,\ \xi(i), \ldots,\ \xi(i_0-1)\ \left(\xi(i_0), \ldots,\ \xi(i_0+m-1)\right), \tag{5}$$

where the sequence enclosed in parenthesis is repeated periodically with period m, will be called a monocyclic sequence [1].

A monocyclic sequence

$$\eta(0), \ldots,\ \eta(i), \ldots,\ \eta(i_0-1)\ \left(\eta(i_0), \ldots,\ \eta(i_0+m-1)\right) \tag{6}$$

with all the first $i_0 + m$ terms distinct will be called a basic sequence.

Any monocyclic sequence can be enlarged to a basic sequence, and obviously this can be done in various ways. In particular, this can be done as follows: We select the smallest number N satisfying the inequality $2^{N-n} \geq M$, where M is the largest number of equal terms among the first $i_o + m$ terms of the sequence (5). Then we replace every term $\xi(i)$ of the sequence (5) by an N-stage binary number $\eta(i)$ of the form (4), where the final (smallest) N-n stages are chosen such that the whole number $\eta_N(i)\, \eta_{N-1}(i) \cdots \eta_{n+1}(i)$ formed of these digits gives the binary representation for the number of preceding terms in the sequence (5) which are equal to the given term $\xi(i)$ of that sequence. The sequence obtained in such a way will not contain the same term twice among its first $i_o + m$ terms, and it will therefore be a basic sequence.

3. A sequence in which every term appears as a well-defined function of a certain number of previous terms of the sequence is called a recurrent sequence. In particular, included among recurrent sequences are all sequences of the form

$$\eta(o) = \eta_o, \quad \eta(i+1) = \phi\left(\eta(i)\right), \tag{7}$$

where ϕ is any well-defined function, η_o is any given number, and i is any positive integer.

From the single-valuedness of the function $\phi(\eta)$, and from the fact that the number of possible terms of the sequence $\eta(i)$ equals 2^n, it follows that every recurring sequence of the form (7) appears as a basic monocyclic sequence with period $m \leq 2^n$.

The converse statement is also correct: Every basic monocyclic sequence (5) appears as a recurring sequence described by the equations (7).

Functions $\phi(\eta)$ can be effectively constructed for any given basic monocyclic sequence (5) with the help of the method described in the following section.

4. Suppose that we regard digits of n-stage binary numbers as components of n-dimensional vectors

$$y(i) = \left[y_1(i), \ldots, y_k(i), \ldots, y_n(i)\right], \tag{8}$$

that is, assume

$$y_k(i) = \eta_k(i) \tag{9}$$

for every k and i,

$$(k = 1, 2, \ldots, n; \quad i = 0, 1, 2, \ldots).$$

Making such a substitution for each term of the given sequence, we obtain the corresponding basic monocyclic sequence of values of the n-dimensional vector y:

$$y(o), \ldots, y(i), \ldots, y(i_o-1) \Big(y(i_o), \ldots, y(i_o+m-1) \Big). \qquad (10)$$

This sequence, as well as the sequence (6) uniquely corresponding to it, appears as a recurring sequence; that is, it satisfies the equations

$$y(o) = y_{j_o},\ y(i+1) = f\big(y(i)\big) \qquad (11)$$

for every positive integer i.

Since the components of the vector y may take only two values, 0 and 1, there exist 2^n distinct values y_j for this vector.

To each value y_j of the vector y, there corresponds bi-uniquely the function

$$p_j(y) = \prod_{k=1}^{n} (y_{j,k} \oplus y_k \oplus 1), \qquad (12)$$

where y_k and $y_{j,k}$ are the k^{th} components of the variable vector y and its corresponding y_j, and the sign $\oplus$ denotes the operation of addition of elements in the field of characteristic 2; that is, the operational equations

$$0 \oplus 0 = 1 \oplus 1 = 0,\quad 0 \oplus 1 = 1 \oplus 0 = 1.$$

The functions $p_j(y)$ in two-element Boolean algebra are called the constituent elements.

In view of the bi-unique correspondence between the values y_j of the vector y and the functions $p_j(y)$, the sequence (10) is equivalent to the following sequence of constituent elements:

$$p_{j_o}(y), \ldots, p_{j_i}(y), \ldots, p_{j_{i_o-1}}(y) \Big(p_{j_{i_o}}(y), \ldots, p_{j_{i_o+m-1}}(y) \Big). \qquad (13)$$

Every vector y from the n-dimensional vector space over the ground field of characteristic 2 will be called an n-dimensional binary vector, and a binary vector appearing as a function of a certain degree, a binary function of that degree.

Any binary function f(y) of the binary vector y can be decomposed into functions $p_j(y)$; that is, a unique representation of the form

$$f(y) = \sum_{j=o}^{2^n-1} f(y_j)\, p_j(y), \qquad (14)$$

where $\sum_{j=0}^{2^n-1}$ is the symbol for the summation of 2^n vectors over the field of characteristic 2. We note that in view of the equation

$$p_j(y)\ p_{j'}(y) = 0, \tag{15}$$

having set $j' \neq j$, and likewise, with $a \cdot b = 0$.

$$a \oplus b = a \vee b, \tag{16}$$

where V is the sign of Boolean alternation, and the sign $\sum$ in formula (14) can be regarded likewise as the sign of Boolean summation. For such purposes the sign $\sum$ in formula (14) can be regarded as a vector form indicating the decomposition of any function of Boolean algebra into constituent elements.

Applying formula (14) to the values of the vector y encountered in the given basic monocyclic sequence (10), we obtain from the basic equation (11) the formula

$$f(y) = \sum_{i=0}^{2^n-1} y(i+1)\ p_{j_i}(y), \tag{17}$$

which allows the determination of the desired function f(y) of the recurring sequence equivalent to the given basic sequence (10), or equivalent finally to the basic monocyclic sequence (6).

In the case that the given monocyclic sequence (5) does not appear to be basic, it is necessary to extend it by any method (for example the method of section 2). to a basic sequence.

The components f_k of the vector function f, computed by formula (17), can be regarded as the corresponding binary digits ϕ_k of the function ϕ in equation (7).

In this form, the method described allows the conversion of any monocyclic sequence of numbers, represented as terminal numbers of binary digits, into recurring sequences, satisfying equation (7). In particular, any sequence of numbers obtained empirically -- for example, any empirical function of one variable -

can be converted by this method into a recurring sequence, described by equation (7).

5. The value of the method of conversion just described is that as a consequence of this method, functions expressed by means of binary digits become terms of a fixed sequence, via the operations of Boolean algebra. This enables every function ϕ resulting from this method to be realizable by means of some autonomous relay system, composed of two-position relays, (By autonomous systems we mean those relay systems which contain no relays controlled from the outside.)

In other words, the method under discussion makes possible the synthesis of autonomous relay systems for any fixed sequence with the states

$$y_{j_0}, y_{j_1}, \ldots, y_{j_i}, \ldots, y_{j_{i_0-1}} (y_{j_{i_0}}, \ldots, y_{j_{i_0+m-1}}) \qquad (18)$$

where y_{j_i} is the initial value of the n-dimensional vector $y(i)$, assuming the condition that all n relays form a relay system for the i^{th} cycle of the operation.

In the case that the initial value $y(0)$ of the relay system is obtained in accordance with the described method, and agrees with one of the terms of the given sequence (18), the process of changing the state of this system becomes identical with that of subsequences of the sequence (18) which begin with a term identical to $y(0)$. In the case that $y(0)$ does not coincide with any term of the sequence (18), this relay system reduces immediately to the null state $y_o = 0$, which is the stable state of the desired system.

The relay systems which can be made in practice by means of some relay-contact system with vectorial properties are characterized as precisely those systems which, for a given vector $y = \{y_1, \ldots, y_k, \ldots, y_n\}$, with components $y_1, \ldots, y_k, \ldots, y_n$, have leads from the values which appear to any closings or openings of contacts with all the relays, to generate the given relay system.

6. The description of the above method can be used for the determination of the function ϕ in equation (7) in technical circumstances, where the values of the functions $\eta(i)$ are determined by these relations. Unrestrained growth may occur, for example, in the case that the values of $\eta(i)$ constitute an arithmetic or geometric progression. The results governing the values of $\eta(i)$ as the final

n binary digits of numbers, may be extended by passage to the limit to the case $n \to \infty$.

This method may be used, for example, to synthesize a primitive recursive function $\beta(i)$ satisfying the conditions [2]

$$\beta(0) = 0, \quad \beta(i+1) = \beta(i) + 1;$$

that is, the function whose sequence of values forms the natural sequence [i.e. the sequence of natural numbers].

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